Time fractional Poisson equations: representation and estimates

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The 14th Workshop on Markov Processes and Related Topics

July 18, 2018

Classical Caputo fractional derivative

Time-fractional equations arise naturally in many fields and from applications. A classical equation of such a kind is

$$\frac{\partial^\beta u(t,x)}{\partial t^\beta} = \Delta u(t,x).$$

Here

$$\begin{aligned} \frac{\partial^\beta \varphi(t)}{\partial t^\beta} &= \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta} \left(\varphi(s) - \varphi(0)\right) ds \\ &= \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \varphi'(s) ds \quad \text{if } \varphi \text{ is Lipschitz,} \end{aligned}$$

where $\Gamma(\lambda) = \int_0^\infty t^{\lambda-1} e^{-t} dt$.

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Fractional-time equation

- Heat propogation in materials having thermal memory
- Subdiffusion describes particle moves slower than Brownian motion, for example, due to particle sticking and trapping.

Example: (i) xerox machine, electrons in amorphous media tend to get trapped by local imperfections and then released due to thermal fluctuations.

(ii) hydrology: travel times of contaminants in groundwater are much longer than that of diffusion.

(iii) biology: proteins diffuse across cell membranes.

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How to model subdiffusions?

A prototype of subdiffusion can be modeled by Brownian motion time-changed by an inverse stable subordinator.

Continuous time random walk model:

$$X_n = \sum_{k=1}^{n} \xi_k, \qquad T_n = \sum_{j=1}^{n} \eta_j,$$

where ξ_k is the kth displacement and η_j is the jth waiting or holding time. Let $N_t = \max\{n : T_n \leq t\}$. Then $Y_t = X_{N_t}$.

Let B is Brownian motion in $\mathbb{R}^d$ and S an β-stable subordinator. Define

$$E_t = \sup\{r > 0 : S_r \leq t\} = \inf\{r > 0 : S_r > t\}.$$

Then B_{E_t} provides a model for anomalous sub-diffusion, where particles spread slower than Brownian particles.

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Fractional-kinetics process

B_{E_t} is called fractional-kinetics process in some literature.

It also arises

(i) (symmetric Bouchaud's trap model) as the quenched scaling limit of random walks in $\mathbb{Z}^d$ with exponential g times at each vertices whose expected values are i.i.d random variables of power law distribution;
Ben Arous-Černý 2007: For $d \geq 3$ and $\beta \in (0,1)$,

$$\left\{ n^{-\beta/2} X_{[nt]}; t \geq 0 \right\} \Rightarrow \{ \mathbf{BM}_{E_t}; t \geq 0 \} .$$

For $d = 2$, the scaling constant is $N^{-\beta/2} (\log N)^{-(1-\beta)/2}$.

(ii) as the quenched scaling limit of constant speed random walks on $\mathbb{Z}^d$ ($d \geq 2$) with i.i.d conductances that have power law tails. (Barlow-Černý 2011 for $d \geq 3$, Černý 2011 for $d = 2$.)

Connection

In general, given a Markov process X_t and an independent β-subordinator S, one can do time change to get a new process $Y_t = X_{E_t}$, where $E_t = \inf\{r \geq 0 : S_r > t\}$.

Question: What is the marginal distribution of Y_t?

Theorem (Baeumer-Meerschaert, 2001; Meerschaert-Scheffler, 2004): $u(t,x) = \mathbb{E}_x[f(X_{E_t})]$ solves

$$\frac{\partial^\beta u(t,x)}{\partial t^\beta} = \mathcal{L}_x u(t,x), \quad u(0,x) = f(x).$$

The self-similarity of the β-subordinator,

$$\{S_{\lambda t}; t \geq 0\} = \{\lambda^{1/\beta} S_t; t \geq 0\} \quad \text{in distribution,}$$

played an important role in their derivation.

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General time-fractional derivative

In applications and numerical approximations, there is a need to consider more general fractional-time derivatives, for example where its value at time t may depend only on the finite range of the past from $t-\delta$ to t such as

$$\frac{d}{dt}\int_{(t-\delta)^+}^{t} (t-s)^{-\beta}\,(f(s)-f(0))\,ds.$$

Given a decreasing function w on $(0,\infty)$ with $\lim_{x\to\infty} w(x)=0$, define

$$\partial_t^w f(t) = \frac{d}{dt}\int_0^t w(t-s)\,(f(s)-f(0))\,ds,$$

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Questions

(i) Existence and uniqueness for solution of

$$(\kappa\partial_t + \partial_t^w)\, u = \mathcal{L}u \quad \text{with } u(0,x) = f(x),$$

and its probabilistic representation.

(ii) Given a strong Markov process X and subordinator S, what equation does $u(t,x) = \mathbb{E}_x\left[f(X_{E_t})\right]$ satisfy? Here

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Subordinator

Given a constant $\kappa \geq 0$ and an unbounded right continuous non-increasing function $w(x)$ on $(0, \infty)$ with $\lim_{x\to\infty} w(x) = 0$ and $\int_0^\infty (1 \wedge x)(-dw(x)) < \infty$, there is a unique subordinator $\{S_t; t \geq 0\}$ with Laplace exponent

$$\phi(\lambda) = \kappa\lambda + \int_0^\infty (1 - e^{-\lambda x})(-dw(x)).$$

Laplace exponent: $\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}$.

Conversely, given a subordinator $\{S_t; t \geq 0\}$, there is a unique constant $\kappa \geq 0$ and a Lévy measure ν on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge x)\nu(dx) < \infty$ so that its Laplace exponent is given by above the display with $w(x) = \nu(x, \infty)$.

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Framework

From now on, we assume S_t is a subordinator with infinite Lévy measure ν and possible drift $\kappa \geq 0$. Define $w(x) = \nu(x, \infty)$.

Facts: Since $\nu(0, \infty) = \infty$, $t \mapsto S_t$ is strictly increasing. Hence the inverse subordinator E_t is continuous in t.

Suppose that $\{T_t; t \geq 0\}$ is a strongly continuous semigroup with infinitesimal generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ in some Banach space $(\mathbb{B}, \|\cdot\|)$ with the property that $\sup_{t>0} \|T_t\| < \infty$. Here $\|T_t\|$ denotes the operator norm of the linear map $T_t : \mathbb{B} \to \mathbb{B}$.

E.g. $(\mathbb{B}, \|\cdot\|) = L^p(E;\mu)$ for $p \geq 1$ or $(C_\infty(E), \|\cdot\|_\infty)$.

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Main result

Theorem (C. 2017)

For every $f \in \mathcal{D}(\mathcal{L})$, $u(t,x) := \mathbb{E}\left[T_{E_t} f(x)\right]$ *is the unique solution in* $(\mathbb{B}, \|\cdot\|)$ *to*

$$(\kappa\partial_t + \partial_t^w)\, u(t,x) = \mathcal{L}u(t,x) \quad \text{with } u(0,x) = f(x).$$

Remarks

(i) The assumption that $f \in \mathcal{D}(\mathcal{L})$ in the Theorem is to ensure that all the integrals involved in the proof are absolutely convergent in the Banach space $\mathbb{B}$. This condition can be relaxed if we formulate the time fractional equation in the weak sense when the uniformly bounded strongly continuous semigroup $\{T_t; t \geq 0\}$ is symmetric in a Hilbert space $L^2(E; m)$ and so its quadratic form can be used to formulate weak solutions. This is done in [CKKW1].

(ii) Special cases or related work: Meerschaert and Scheffler (2008) and Kolokoltsov (2011). (Toaldo (2015)).

(iii) There are very limited results on uniqueness.

(iv) One needs to be very careful when dealing with time frictional equations due to nature of singular integrals. Probabilistic representation turns out to be quite effective to overcome these difficulties.

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Fundamental solution

When the uniformly bounded strongly continuous semigroup $\{T_t; t \geq 0\}$ has an integral kernel $p_0(t, x, y)$ with respect to some measure $m(dx)$, then there is a kernel $p(t, x, y)$ so that

$$u(t, x) := \mathbb{E}\left[T_{E_t} f(x)\right] = \int_E p(t, x, y) f(y) m(dy);$$

in other words,

$$p(t, x, y) := \mathbb{E}\left[p_0(E_t, x, y)\right] = \int_0^\infty p_0(s, x, y) d_s \mathbb{P}(E_t \leq s)$$

is the fundamental solution to the time fractional equation $(\kappa \partial_t + \partial_t^w) u = \mathcal{L} u$.

In a recent work with Kim, Kumagai and Wang, two-sided estimates on $p(t, x, y)$ are obtained when $\kappa = 0$ and $\{T_t; t \geq 0\}$ is the transition semigroup of a diffusion process that satisfies two-sided Gaussian-type estimates or of a stable-like process on metric measure spaces.

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Let $0 < \beta < 1$. How to solve

$$\partial_t^\beta u(t,x) = \Delta u(t,x) + f(t,x)$$

with $u(0,x) = 0$?

Let $p(t,x,y) = \mathbb{E} p_0(E_t, x, y)$ be the fundamental solution of $\partial_t^\beta u(t,x) = \Delta u(t,x)$. Define

$$q(t,x,y) = \partial_t^{1-\beta} p(\cdot, x, y)(t).$$

It is known in literature that

$$u(t,x,y) = \int_0^t \int_{\mathbb{R}^d} q(s,x,y) f(t-s,y) dyds$$

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Poisson equations

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Questions

• Solution in which sense?

• How to verify $t \to p(t, x, y)$ is time-fractional differentiable?

• Positivity: If $f(t, x, y) \geq 0$, is the solution $u(t, x) \geq 0$?

• What happens for general spatial generator $\mathcal{L}$ and for general time fractional derivatives?

Set up

Assume that $\{S_t, \mathbb{P}; t \geq 0\}$ is a driftless subordinatorwith infinite Lévy measure ν and having bounded density $\overline{p}(r, \cdot)$ for each $r > 0$. A sufficient condition for the latter is

$$\lim_{s\to\infty} \frac{\phi(s)}{\ln(1+s)} = \lim_{s\to\infty} \frac{1}{\ln(1+s)} \int_0^\infty (1 - e^{-sx})\, \nu(dx) = \infty.$$

(Hartman and Wintner's condition.)

Suppose that $\{P_t^0;\ t \geq 0\}$ is a uniformly bounded strongly continuous semigroup in some Banach space $(\mathbb{B}, \|\cdot\|)$ and $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is its infinitesimal generator.

Goal: For any $T_0 > 0$, solve

$$\partial_t^w u = \mathcal{L}u + f(t,x) \quad \text{on } (0, T_0] \times E$$

with $u(0,x) = g(x)$.

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Theorem (C.-Kim-Kumagai-Wang, 2018+)

Let $g \in \mathcal{D}(\mathcal{L})$ and $f(t,x)$ be a function defined on $(0, T_0] \times E$ so that for a.e. $t \in (0, T_0]$, $f(t, \cdot) \in \mathcal{D}(\mathcal{L})$ and

$$\text{esssup}_{t\in[0,T_0]} \|f(t,\cdot)\| + \int_0^{T_0} \|\mathcal{L}f(t,\cdot)\| dt < \infty.$$

The function

$$\begin{aligned} u(t,x) &= \mathbb{E}\left[P^0_{E_t} g(x)\right] + \mathbb{E}\left[\int_0^\infty 1_{\{S_r<t\}} P^0_r f(t-S_r,\cdot)(x) dr\right] \\ &= \mathbb{E}\left[P^0_{E_t} g(x)\right] + \int_{s=0}^t \int_{r=0}^\infty P^0_r f(t-s,\cdot)(x)\overline{p}(r,s)\, dr\, ds \end{aligned}$$

is the unique (strong) solution of $\partial^w_t u = \mathcal{L}u + f(t,x)$ on $(0, T_0] \times E$ with $u(0,x) = g(x)$ in the following sense.

Poisson equation

Theorem (C.-Kim-Kumagai-Wang, 2018+)

(i) $u(t, \cdot)$ *is well defined as an element in* $\mathbb{B}$ *for each* $t \in (0, T_0]$ *such that* $\sup_{t\in(0,T_0]} \|u(t,\cdot)\| < \infty$, $t \mapsto u(t,x)$ *is continuous in* $(\mathbb{B}, \|\cdot\|)$ *and* $\lim_{t\to 0} \|u(t,\cdot) - g\| = 0$.

(ii) *For a.e.* $t \in (0, T_0]$, $u(t,\cdot) \in \mathcal{D}(\mathcal{L})$ *and* $\mathcal{L}u(t,\cdot)$ *exists in the Banach space* $\mathbb{B}$ *with* $\int_0^{T_0} \|\mathcal{L}u(t,\cdot)\| dt < \infty$.

(iii) *For every* $T \in (0, T_0]$,

$$\int_0^T w(T-t)(u(t,\cdot) - g)\, dt = \int_0^T (f(t,\cdot) + \mathcal{L}u(t,\cdot))\, dt \quad \text{in } \mathbb{B}.$$

We also have corresponding result for weak solutions.

Poisson equation

Theorem (C.-Kim-Kumagai-Wang, 2018+)

(i) $u(t,\cdot)$ *is well defined as an element in* $\mathbb{B}$ *for each* $t \in (0, T_0]$ *such that* $\sup_{t\in(0,T_0]} \|u(t,\cdot)\| < \infty$, $t \mapsto u(t,x)$ *is continuous in* $(\mathbb{B}, \|\cdot\|)$ *and* $\lim_{t\to 0} \|u(t,\cdot) - g\| = 0$.

(ii) *For a.e.* $t \in (0, T_0]$, $u(t,\cdot) \in \mathcal{D}(\mathcal{L})$ *and* $\mathcal{L}u(t,\cdot)$ *exists in the Banach space* $\mathbb{B}$ *with* $\int_0^{T_0} \|\mathcal{L}u(t,\cdot)\| dt < \infty$.

(iii) *For every* $T \in (0, T_0]$,

$$\int_0^T w(T-t)(u(t,\cdot)-g)\, dt = \int_0^T (f(t,\cdot) + \mathcal{L}u(t,\cdot))\, dt \quad \text{in } \mathbb{B}.$$

We also have corresponding result for weak solutions.

Another fundamental solution

Suppose that $(\mathbb{B}, \|\cdot\|) = L^p(E;\nu)$ or $C_\infty(E)$, and the semigroup $\{P_t^0; t \geq 0\}$ has an integrable kernel $p_0(t,x,y)$ with respect to some measure $\mu(dx)$ on E. Define

$$q(t,x,y) = \int_0^\infty p_0(r,x,y)\overline{p}(r,t)dr.$$

Then the unique solution in above theorem can be expressed as

$$u(t,x) = \int_E p(t,x,y)g(y)\mu(dy) + \int_0^t \int_E q(s,x,y)f(t-s,y)\mu(dy)ds.$$

(Recall $p(t,x,y) = \mathbb{E}\left[p_0(E_t,x,y)\right]$.)

Remarks

• Positivity of $q(t, x, y)$.

• Two-sided estimates of $q(t, x, y)$.

• Stability of $p(t, x, y)$ and $q(t, x, y)$.

• An analogous probabilistic representation for solutions of Poisson equation has been obtained recently by M. E. Hernández-Hernández, V. N. Kolokoltsov and L. Toniazzi (2017) and L. Toniazzi (2018) using a different approach and in restrictive settings (Caputo derivative in time and Feller generator $\mathcal{L}$ in space $\mathbb{R}^d$, using Mittag-Leffer functions).

A connection

Suppose S is a special Bernstein function; that is, $\lambda \mapsto \lambda/\phi(\lambda)$ is still a Bernstein function. Let S^* be the subordinator with Laplace expondent $\lambda/\phi(\lambda)$. Suppose S_t has density function $p(r,t)$.

Theorem (C.-Kim-Kumagai-Wang, 2018+)

For a.e. $x \neq y \in E$,

$$q(t,x,y) = \partial_t^{w^*} p(\cdot,x,y)(t)$$

in the sense that for all $t > 0$,

$$\int_0^t q(s,x,y)ds = \int_0^t w^*(t-s)p(s,x,y)ds.$$

Heat kernel estimates for $\mathcal{L}$

$$p_0(t,x,y) \asymp t^{-d/\alpha} F(d(x,y)/t^{1/\alpha}).$$

1) $F(r) = \exp\left(-r^{\alpha/(\alpha-1)}\right)$ for $\alpha \geq 2$: local case

• $\alpha = 2$ when $\mathcal{L} = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$ with $\lambda^{-1} I \leq (a_{ij}(x)) \leq \lambda I$ on $\mathbb{R}^d$; Aronson 1967

• $\alpha > 2$ when $\mathcal{L}$ is the Laplacian on Sierpinski gasket or carpet; Barlow-Perkins 1988, Barlow-Bass 1992. E.g. two-dimensional Sierpinski gasket, $d = \log 3/\log 2$ and $\alpha = d_w := \log 5/\log 2$.

2) $F(r) = (1+r)^{-d-\alpha}$ with $\alpha > 0$: non-local case:

• symmetric stable-like process on Alfhors d-regular space E. C.-Kumagai 2003 ($\alpha < 2$), C.-Kumagai-Wang 2018 ($\alpha < d_w$):

$$\mathcal{L}f(x) = \mathrm{p.v.}\int_E (f(y) - f(x))\frac{c(x,y)}{|x-y|^{d+\alpha}}\mu(dy).$$

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Estimate

Particular case: $S_t = \beta$-subordinator, or Caputo derivative ∂_t^β.

Define

$$\widetilde{H}_{\leq 1}(t, d(x,y)) = \begin{cases} t^{\beta-1-\beta d/\alpha}, & d < 2\alpha, \\ t^{-\beta} \log\left(\frac{2t^\beta}{d(x,y)^\alpha}\right), & d = 2\alpha, \\ = t^{1-\beta}/d(x,y)^{d-2\alpha}, & d > 2\alpha, \end{cases}$$

$$\widetilde{H}^{(c)}_{\geq 1}(t, d(x,y)) = t^{\beta-1-\beta d/\alpha}\left(d(x,y)^\alpha/t^\beta\right)^{(1-\beta)/(\alpha-\beta)} \times \exp\left(-\left(d(x,y)^\alpha/t^\beta\right)^{1/(\alpha-\beta)}\right),$$

$$\widetilde{H}^{(j)}_{\geq 1}(t, d(x,y)) = t^{2\beta-1}/d(x,y)^{d+\alpha}.$$

Estimates of fundamental solution

Theorem (C.-Kim-Kumagai-Wang 2018+)

(i) *Suppose* $F(r) = \exp(-r^{\alpha/(\alpha-1)})$ *with* $\alpha \geq 2$. *Then*

$$q(t,x,y) \simeq \widetilde{H}_{\leq 1}(t, d(x,y)) \quad \text{if } d(x,y) \leq t^{\beta/\alpha},$$
$$q(t,x,y) \asymp \widetilde{H}^{(c)}_{\geq 1}(t, d(x,y)) \quad \text{if } d(x,y) \geq t^{\beta/\alpha}.$$

(ii) *Suppose* $F(r) = (1+r)^{-d-\alpha}$. *Then,*

$$q(t,x,y) \simeq \widetilde{H}_{\leq 1}(t, d(x,y)) \quad \text{if } d(x,y) \leq t^{\beta/\alpha},$$
$$q(t,x,y) \simeq \widetilde{H}^{(j)}_{\geq 1}(t, d(x,y)) \quad \text{if } d(x,y) \geq t^{\beta/\alpha}.$$

Approach

We mainly use probabilistic approach by studying the properties of subordinator and inverse subordinator. Here is an example: How did we get

$$q(t,x,y) := \partial_t^{w^*} p(t,x,y) = \int_0^\infty p_0(r,x,y)\overline{p}(r,t)dr?$$

First for $t > 0$, define $G^*(t) := \int_0^t w^*(r)dr$.

Lemma (C. 2017)

For every $t > 0$,

$$\int_0^t w^*(t-s)\mathbb{P}(S_r > s)\, ds = G^*(t) - \mathbb{E}\left[G^*(t-S_r)1_{\{t\geq S_r\}}\right].$$

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Approach: Poisson equation

For any $x \neq y$, since $p(0,x,y) = 0$,

$$\begin{aligned}
\partial_t^{w^*} p(t,x,y) &= \frac{d}{dt}\int_0^t w^*(t-s)p(s,x,y)\,ds \\
&= \frac{d}{dt}\int_0^t w^*(t-s)\int_0^\infty p_0(r,x,y)\,d_r\mathbb{P}(E_s \le r)\,ds \\
&= \frac{d}{dt}\int_0^\infty p_0(r,x,y)\,d_r\left(\int_0^t w^*(t-s)\mathbb{P}(S_r \ge s)\,ds\right) \\
&= \frac{d}{dt}\int_0^\infty p_0(r,x,y)\,d_r\left(G^*(t) - \mathbb{E}(G^*(t-S_r)1_{\{t\ge S_r\}})\right) \\
&= -\frac{d}{dt}\int_0^\infty p_0(r,x,y)\,d_r\mathbb{E}(G^*(t-S_r)1_{\{t\ge S_r\}}) \\
&= -\frac{d}{dt}\int_0^\infty p_0(r,x,y)\,P_r^S\left(\mathcal{L}_S G^*(t-\cdot)1_{\{t\ge \cdot\}}\right)(0)dr.
\end{aligned}$$

Approach: Poisson equation

For any $x \le t$,

$$\begin{aligned}
&\mathcal{L}_S\left(G^*(t-\cdot)1_{\{t\ge\cdot\}}\right)(x) \\
&= \int_0^\infty \left(G^*((t-x-z)^+) - G^*(t-x)\right)\nu(dz) \\
&= \cdots\cdots \qquad \text{(using Fubini)} \\
&= -\int_0^{t-x} w(t-x-r)w^*(r)\,dr = -1
\end{aligned}$$

Clear that $\mathcal{L}_S G(t-\cdot)(x) = 0$ if $x > t$. Therefore,

$$\begin{aligned}
\partial_t^{w^*} p(t,x,y) &= \frac{d}{dt}\int_0^\infty p_0(r,x,y)P_r^S 1_{(0,t]}(0)\,dr \\
&= \int_0^\infty p_0(r,x,y)\overline{p}(r,t)\,dr \quad \text{if } S_r \text{ has a density } \overline{p}(r,t).
\end{aligned}$$

But ...

The function $G^*(t-\cdot)1_{\{t\geq\cdot\}}$ is not in the domain of the Feller generator of the subordinator, and the last equality needs justification. Thus we need a different proof.

Our final proof in fact goes the other way around. We define

$$q(t,x,y)=\int_0^\infty p_0(r,x,y)\overline{p}(r,t)\,dr$$

and show that it is the fundamental solution for Poisson equation

$$\partial_t^w u=\mathcal{L}u+f(t,x)\quad \text{with } u(0,x)=0.$$

We then show that if the subordinator is special, then $q(t,x,y)=\partial_t^{w^*}p(\cdot,x,y)(t)$ using Laplace transform.

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Examples

(i) When $\{S_t; t \geq 0\}$ is a β-subordinator with $0 < \beta < 1$ with Laplace exponent $\phi(\lambda) = \lambda^\beta$, Then S_t has no drift (i.e. $\kappa = 0$) and its Lévy measure is $\mu(dx) = \frac{\beta}{\Gamma(1-\beta)} x^{-(1+\beta)} dx$. Hence

$$w(x) := \mu(x, \infty) = \int_x^\infty \frac{\beta}{\Gamma(1-\beta)} y^{-(1+\beta)} dy = \frac{x^{-\beta}}{\Gamma(1-\beta)}.$$

Thus the time fractional derivative $\partial_t^w f$ is exactly the Caputo derivative of order β. In this case, our Theorem recovers the main result of Baeumer-Meerschaert (2001) and Meerschaert-Scheffler (2004).

Truncated stable-subordinator

(ii) A truncated β-stable subordinator $\{S_t; t \geq 0\}$ is driftless and has Lévy measure

$$\mu_\delta(dx) = \frac{\beta}{\Gamma(1-\beta)} x^{-(1+\beta)} 1_{(0,\delta]}(x) dx$$

for some $\delta > 0$. In this case,

$$\begin{aligned} w_\delta(x) &:= \mu_\delta(x,\infty) = 1_{\{0<x\leq\delta\}} \int_x^\delta \frac{\beta}{\Gamma(1-\beta)} y^{-(1+\beta)} dy \\ &= \frac{1}{\Gamma(1-\beta)} \left(x^{-\beta} - \delta^{-\beta} \right) 1_{(0,\delta]}(x). \end{aligned}$$

The corresponding the fractional derivative is

$$\partial_t^{w_\delta} f(t) := \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_{(t-\delta)^+}^t \left((t-s)^{-\beta} - \delta^{-\beta} \right) (f(s) - f(0)) ds.$$

Stability

Clearly, as $\lim_{\delta\to\infty} w_\delta(x) = w(x) := \frac{1}{\Gamma(1-\beta)} x^{-\beta}$. Consequently, $\partial_t^{w_\delta} f(t) \to \partial_t^w f(t)$, the Caputo derivative of f of order β, in the distributional sense as $\delta \to 0$. Using the probabilistic representation in the main Theorem, one can deduce that as $\delta \to \infty$, the solution to the equation $\partial_t^{w_\delta} u = \mathcal{L} u$ with $u(0,x) = f(x)$ converges to the solution of $\partial_t^\beta u = \mathcal{L} u$ with $u(0,x) = f(x)$.

(iii) If we define

$$\eta_\delta(r) = \frac{\Gamma(2-\beta)\,\delta^{\beta-1}}{\beta} w_\delta(r) = (1-\beta)\delta^{\beta-1}\left(x^{-\beta} - \delta^{-\beta}\right) 1_{(0,\delta]}(x),$$

then $\eta_\delta(r)$ converges weakly to the Dirac measure concentrated at 0 as $\delta \to 0$. So $\partial_t^{\eta_\delta} f(t)$ converges to $f'(t)$ for every differentiable f. It can be shown that the subordinator corresponding to η_δ, that is, subordinator with Lévy measure

$$\nu_\delta(dx) := \frac{(1-\beta)\,\delta^{\beta-1}}{\beta} x^{-(1+\beta)} 1_{(0,\delta]}(x) dx,$$

converges as $\delta \to 0$ to deterministic motion t moving at constant speed 1. Using the main Theorem, one can show that the solution to the equation $\partial_t^{\eta_\delta} u(t,x) = \mathcal{L}u(t,x)$ with $u(0,x) = f(x)$ converges to the solution of the heat equation $\partial_t u = \mathcal{L}u$ with $u(0,x) = f(x)$.

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Thank you!